N_T = number of theoretical units
P_0 = partial pressure of adsorbate
P_v = vapor pressure of adsorbate
R = gas constant
T = temperature
U = solute adsorbed in adsorption zone
w_a = quantity of effluent from adsorber between breakthrough and exhaustion
w_e = quantity of effluent from adsorber at breakthrough
X_u = adsorbent composition at exhaustion
X_T = adsorbent composition at total saturation
x = volume of HF adsorbed
Y_0 = adsorbate composition of entering gas
Y_B = adsorbate composition of gas at breakthrough
Z = height of adsorbent bed
Z_a = height of exchange zone
ϵ = adsorption potential
ϕ = volume of adsorption
ρ = density at boiling point

LITERATURE CITED

Baker, B. B., and J. D. Morrison, "Determination of Microgram Quantities of Fluoride and Cyanide by Measurement of Current from Spontaneous Electrolysis," *Anal. Chem.*, **27**, 1306 (1955).

Berenyi, L., "Nevere Adsorptionsberechnugen," *Z. Phys. Chem.*, **105**, 55 (1923).

Cochran, C. N., W. C. Sleppy, and W. B. Frank, "Fumes in Aluminum Smelting," *J. Metals*, **22**, 54 (1970).

Cook, C. C., G. R. Swany, and J. W. Colpitts, "Operating Experience with the Alcoa 398 Process for Fluoride Recovery," *J. Air Pollut. Control*, **21**, 479 (1971).

Michaels, A. S., "Simplified Method of Interpreting Kinetic Data in Fixed Bed Ion Exchange," *Ind. Eng. Chem.*, **44**, 1922 (1962).

Perry, J. H. (Ed.) *Chemical Engineers Handbook*, p. **16**-12, 4th Ed., McGraw Hill, N. Y., 1964.

Treybal, R. E., *Mass Transfer Operations*, p. 550, McGraw Hill, N. Y. (1968).

Manuscript received October 12, 1972; revision received November 13, 1972; note accepted November 14, 1972.

Suboptimal Control of Stochastic Distributed Parameter Systems

T. K. YU and J. H. SEINFELD

Department of Chemical Engineering
California Institute of Technology, Pasadena, California 91109

A problem with important process applications is the control of distributed parameter systems subject to boundary and volume disturbances and measurement errors. The optimal control of linear stochastic distributed systems has been studied by Tzafestas and Nightingale (1968), Kushner (1968), Thau (1969), Pell and Aris (1970), Bensoussian (1971), and Sholar and Wiberg (1972), while feedback control of nonlinear stochastic distributed systems was investigated by Yu and Seinfeld (1972). The objective of this communication is to propose a suboptimal scheme for control of nonlinear noisy distributed parameter systems. Briefly, the scheme is based on the extension of the Lyapunov functional method and is designed to be easy to implement in practice.

STATEMENT OF THE PROBLEM

We consider the class of systems governed by the vector nonlinear partial differential equation

$$x_t(r, t) = f[r, t, x, x_r, x_{rr}, v(r, t)] + \xi(r, t) \quad (1)$$

defined for time $t \geq 0$ on the normalized spatial domain $r \epsilon (0, 1)$. $x(r, t)$ is the n-vector state, x_t, x_r, x_{rr} denote $\partial x/\partial t$, $\partial x/\partial r$ and $\partial^2 x/\partial r^2$, respectively, and $v(r, t)$ is a known p-vector ($p \leq n$) volume control. The boundary conditions for Equation (1) are given by the l-vector ($l \leq n$) functions

$$g_0(t, x, x_r, \omega_0(t)) + \xi_0(t) = 0 \quad r = 0 \quad (2)$$

$$g_1(t, x, x_r, \omega_1(t)) + \xi_1(t) = 0 \quad r = 1 \quad (3)$$

where $\omega_0(t)$ and $\omega_1(t)$ are known p_0 and p_1-vector boundary controls, respectively.

Observations of the system are made continuously in time at N discrete spatial positions in the form

$$y(r_i, t) = h(r_i, t, x(r_i, t)) + \eta(r_i, t) \qquad i = 1, 2, \ldots, N \quad (4)$$

where $y(r_i, t)$, the observation, is an m-vector ($m \leq n$) and $r_i \epsilon [0, 1]$.

The random disturbances $\xi(r, t)$, $\xi_0(t)$, $\xi_1(t)$ and $\eta(r_i, t)$ are assumed to have unknown statistical properties other than they are independent and have zero mean.

An important general class of control problems is the regulator problem. The general regulator problem for the system (1) to (4) would involve keeping the state $x(r, t)$ as close as possible to some desired state $x^d(r)$ over some period of time $[0, t_f]$ on the basis of the observations $y(r_i, t)$, $i = 1, 2, \ldots, N$. Formally stated, we want to specify the volume control $v(r, t)$ and the boundary controls, $\omega_0(t)$ and $\omega_1(t)$ as functions of the observations, to minimize the performance index

$$J = E\left\{ \int_0^{t_f} \int_0^1 \int_0^1 <(x(r,t) - x^d(r)), W(r,s,t)\ (x(s,t) - x^d(s))> dr\, ds\, dt \right\} \quad (5)$$

where $W(r, s, t)$ is a positive-definite weighting matrix, for which $W(r, s, t) = W^T(s, r, t)$, and $E(\cdot)$ denotes the expected value. Also, $<a, b>$ denotes the norm a^Tb. Because of the intractability of this performance index, we will shortly propose a related index to replace Equation (5).

THE SUBOPTIMAL SCHEME

The solution to this problem, and indeed to the corresponding deterministic problem, cannot be obtained in the nonlinear case. Thus, we must resort to a suboptimal solution. It is reasonable to sacrifice some degree of optimality to achieve a control scheme that is both simple and practical. We set the following ground rules:

1. The control must be easy to implement.
2. The control will be based on the state estimates $\hat{x}(r, t)$ rather than on the observations $y(r_i, t)$. Thus, a filter will be required in the control loop.

We will consider only a special class of the system (1)-(4), namely where the control is exercised through a volume control $v(r, t)$ and where the control $v(r, t)$ appears linearly on the right-hand side of Equation (1),

$$f(r, t, x, x_r, x_{rr}, v) = \phi(r, t, x, x_r, x_{rr}) + B(r, t, x)\ v(r, t) \quad 6)$$

ϕ is an n-vector function and B is an $n \times p$ matrix.

We propose to design the controller to achieve an instantaneous rather than an overall optimum. This approach has been employed in deterministic distributed parameter control problems by Paradis and Perlmutter (1966) and Vermeychuk and Lapidus (1972). An approach for stochastic lumped parameter systems similar to what we will propose here was presented by Seinfeld (1970).

It is not practical to have the control $v(r, t)$ vary continuously along r. Therefore, we propose to have $v(r, t)$ be piecewise constant over (0, 1). In particular, let the spatial interval (0, 1) be divided into α disjoint subintervals F_i, $i = 1, 2, \ldots, \alpha$, such that $(0, 1) = \bigcup_{i=1}^{\alpha} F_i$. At any time t we require that the control $v(r, t)$ be constant over each subinterval F_i. We let

$$v(r, t) = v(a_i, t)\ r\, \epsilon F_i \quad i = 1, 2, \ldots, \alpha \quad (7)$$

The $v(a_i, t)$ are to be determined as follows. Choose α discrete spatial points, $a_i\, \epsilon F_i$, $i = 1, 2, \ldots, \alpha$, and define an instantaneous criterion of system deviation from the desired state at the α points by

$$J = \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} <(\hat{x}(a_i, t) - x^d(a_i)), W_{ij}(\hat{x}(a_j, t) - x^d(a_j))> \quad (8)$$

where $\hat{x}(a_i, t)$ is the filter estimate and $x^d(a_i)$ the desired value. Note that the spatial positions a_i, $i = 1, 2, \ldots, \alpha$, upon which the control is based need not be the same as those at which the measurements are made, r_i, $i = 1, 2, \ldots N$. The matrix W_{ij} is taken to symmetric and positive-definite.

We propose to choose $v(a_i, t)$, $i = 1, 2, \ldots, \alpha$, to maximize the rate of movement of $\hat{x}(a_i, t)$ towards $x^d(a_i)$. This can be achieved if $\dot{J}$ is minimized at each instant of time with respect to $v(a_i, t)$. Since the amount of control we can apply will almost always be bounded, we impose upper and lower bounds on each $v(a_i, t)$,

$$v_*(a_i) \leq v(a_i, t) \leq v^*(a_i) \quad i = 1, 2, \ldots, \alpha \quad (9)$$

Differentiating J with respect to t gives

$$\dot{J} = 2 \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} <\dot{\hat{x}}(a_i, t), W_{ij}(\hat{x}(a_j, t) - x^d(a_j))> \quad (10)$$

We will want to minimize $\dot{J}$ at each instant of time. In order to do this, we first need to investigate the estimate $\hat{x}(a_i, t)$.

NONLINEAR FILTER

We note that we need the state estimates $\hat{x}(r, t)$. These are to be provided by a nonlinear filter. A general nonlinear filter which provides estimates $\hat{x}(r, t)$ of $x(r, t)$ based on the noisy observations $y(r_i, t)$ has been derived by Seinfeld et al. (1971) and Hwang et al. (1972). The filter consists of the following equations:

$$\hat{x}_t(r, t) = f(r, t, \hat{x}, \hat{x}_r, \hat{x}_{rr}, v) + \sum_{i=1}^{N} \sum_{j=1}^{N} P(r, r_i, t) h_x^T(r_i, t, \hat{x}) Q_{ij}(t) \cdot [y(r_j, t) - h(r_j, t, \hat{x})] \quad (11)$$

$$g_0(t, \hat{x}, \hat{x}_r, \omega_0) = 0 \quad r = 0 \quad (12)$$

$$g_1(t, \hat{x}, \hat{x}_r, \omega_1) = 0 \quad r = 1 \quad (13)$$

$$P_t(r, s, t) = \hat{f}_x(r) P(r, s, t) + P(r, s, t) \hat{f}_x^T(s) + \hat{f}_{x_r}(r) P_r(r, s, t) + P_s(r, s, t) \hat{f}_{x_s}^T(s) + \hat{f}_{x_{rr}}(r) P_{rr}(r, s, t) + P_{ss}(r, s, t) \hat{f}_{x_{ss}}^T(s) + \sum_{i=1}^{N} \sum_{j=1}^{N} P(r, r_i, t)\, S_{rj}(r_i, r_j, t) P(r_j, s, t) + \sum_{i=1}^{N} \sum_{j=1}^{N} P(r, r_i, t)\, S_{ri}(r_i, r_j, t) P(r_i, s, t) + R^+(r, s, t) \quad (14)$$

$$P(r, s, 0) = P_0(r, s) \quad (15)$$

$$\hat{g}_{0_x} P(r, s, t) + \hat{g}_{0_{x_r}} P_r(r, s, t) + R_0^{-1}(t)\, \hat{g}_{0_{x_r}}^{-1} \hat{f}_{x_{rr}}^T\ \delta(s) = 0, \quad r = 0 \quad (16)$$

$$\hat{g}_{1_x} P(r, s, t) + \hat{g}_{1_{x_r}} Pr(r, s, t) - R_1^{-1}(t)\, \hat{g}_{1_{x_r}}^{-1} \hat{f}_{x_{rr}}^T\ \delta(s - 1) = 0, \quad r = 1 \quad (17)$$

where the $n \times n$ matrices

$$S_{r_j}(r_i, r_j, t) = \frac{\partial H(r_i, r_j, t)}{\partial \hat{x}(r_j, t)}$$

$$S_{r_i}(r_i, r_j, t) = \frac{\partial H(r_i, r_j, t)}{\partial x(r_i, t)} \quad (18)$$

and the n-vector

$$H(r_i, r_j, t) = h_x^T(r_i, t, \hat{x}) \\ Q_{ij}(t)\,[y(r_j, t) - h(r_j, t, \hat{x})] \quad (19)$$

The weighting matrices $Q_{ij}(t)$, $R(r, s, t)$, $R_0(t)$ and $R_1(t)$ are continuous with respect to their arguments and positive-definite with the symmetrical properties: $Q_{ij}(t) = Q_{ji}^T(t)$, $R(r, s, t) = R^T(s, r, t)$, $R_0(t) = R_0^T(t)$ and $R_1(t) = R_1^T(t)$. Furthermore, we assume the existence of the generalized inverse matrix $R^+(r, s, t)$ satisfying

$$\int_0^1 R^+(r, \rho, t)\, R(\rho, s, t)\, d\rho = I\,\delta(r - s) \quad (20)$$

where $\delta(\cdot)$ is the Dirac delta and I is the identity matrix. We assume that the inverses of $\hat{g}_{0_{x_r}}$ and $\hat{g}_{1_{x_r}}$ exist when these are square matrices. If not square, $\hat{g}^{-1}_{0_{x_r}}$ and $\hat{g}^{-1}_{1_{x_r}}$ are to be interpreted as the left pseudo inverses

$$\hat{g}^{-1}_{0_{x_r}} = (\hat{g}_{0_{x_r}} \hat{g}^T_{0_{x_r}})^{-1} \hat{g}_{0_{x_r}}$$

$$\hat{g}^{-1}_{1_{x_r}} = (\hat{g}_{1_{x_r}} \hat{g}^T_{1_{x_r}})^{-1} \hat{g}_{1_{x_r}} \quad (21)$$

If $\hat{g}_{0_{x_r}} = 0$ and $\hat{g}^{-1}_{0_x}$ exists, the third term of Equation (16) should be replaced by $R_0^{-1}(t)\ \hat{g}^{-1}_{0_x} \hat{f}^T_{x_{rr}}\ \delta'(s)$, where $\delta'(s)$ denotes $d\delta(s)/ds$ (Carrier et al., 1966). If $\hat{g}_{1_x} = 0$ and $\hat{g}^{-1}_{1_x}$ exists, the third term of Equation (17) should be replaced by $-R_1^{-1}(t)\ \hat{g}^{-1}_{1_x} \hat{f}^T_{x_{rr}}\ \delta'(s - 1)$. Boundary conditions at $s = 0$ and $s = 1$ on $P(r, s, t)$ follow from the symmetry property $P(r, s, t) = P^T(s, r, t)$.

MINIMIZATION OF $\dot{J}$

From Equations (6) and (11) we let $\dot{J} = J_1 + J_2$, where

$$J_1 = 2 \sum_{i=1}^{\alpha} \sum_{j=1}^{\alpha} <\gamma(a_i, t), \\ W_{ij}(\hat{x}(a_j, t) - x^d(a_j))> \quad (22)$$

$$J_2 = 2 \sum_{i=1}^{\alpha} <v(a_i, t), \beta(a_i t)> \quad (23)$$

and the n-vector $\gamma(a_i, t)$ and the p-vector $\beta(a_i, t)$ are defined by

$$\gamma(a_i, t) = \phi(a_i, t, \hat{x}, \hat{x}_{a_i}, \hat{x}_{a_i a_i}) \\ + \sum_{l=1}^{N} \sum_{k=1}^{N} P(a_i, r_l, t) h_x^T(r_l, t, \hat{x})\, Q_{lk}(t). \\ [y(r_k, t) - h(r_k, t, \hat{x})] \quad (24)$$

$$\beta(a_i, t) = B^T(a_i, t, \hat{x}) \sum_{j=1}^{\alpha} W_{ij}(\hat{x}(a_j, t) - x^d(a_j)) \quad (25)$$

It is easy to see that $\dot{J}$ is minimized if

$$v_k(a_i, t) = \begin{cases} v^*_k(a_i) & \beta_k(a_i, t) < 0 \\ v_{k*}(a_i) & \beta_k(a_i, t) > 0 \end{cases} \quad \begin{array}{l} i = 1, 2, \ldots, \alpha \\ k = 1, 2, \ldots, p \end{array} \quad (26)$$

If $\beta_k(a_i, t) = 0$, we can hold $v_k(a_i, t)$ at its previous value until $\beta_k \neq 0$. Since we are not obtaining the true optimal solution, $\beta_k = 0$ over an interval of time cannot be construed as a singular interval.

A NUMERICAL EXAMPLE

We consider the following scalar, linear parabolic system ($n = 1, p = 1, m = 1$),

$$x_t(r, t) = 0.5\, x_{rr}(r, t) + v(r, t) \quad (27)$$

$$x_r = 0.8 + \xi_0(t) \quad r = 0 \quad (28)$$

$$x_r = 0 \quad r = 1 \quad (29)$$

$$y(r_i, t) = x(r_i, t) + \eta(r_i, t) \quad i = 1, 2, 3 \quad (30)$$

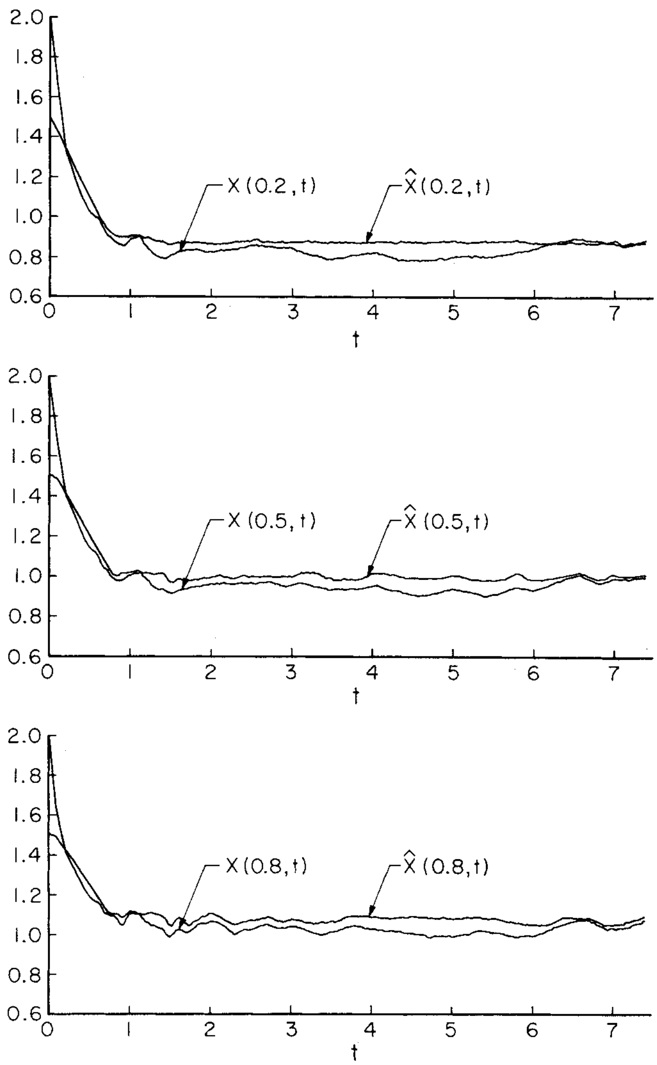

Fig. 1. Filter performance.

where the unknown initial state is $x(r, 0) = 1.5$. The observations are made at $r_1 = 0.2$, $r_2 = 0.5$ and $r_3 = 0.8$. It is desired to drive the system to the desired state, $x^d(a_1) = 0.9$, $x^d(a_2) = 1.0$ and $x^d(a_3) = 1.1$, where $a_1 = 0.2$, $a_2 = 0.5$ and $a_3 = 0.8$. The performance index is

$$J = \sum_{i=1}^{3} (\hat{x}(a_i, t) - x^d(a_i))^2 \tag{31}$$

We assume that the control can be exercised only in the region (0.05, 0.95). The design of the control is $v(r, t) = v(a_i, t)$, $r \epsilon F_i$, $F_1 = (0.05, 0.35)$, $F_2 = [0.35, 0.65]$, and $F_3 = (0.65, 0.95)$, with $v(r, t) = 0$, $r \not\epsilon F_i$. The control bounds were chosen arbitrarily as

$$v^*(a_1) = 0.75 \quad v^*(a_2) = 0.75 \quad v^*(a_3) = 1.0$$

$$v_*(a_1) = -0.25 \quad v_*(a_2) = -0.375 \quad v_*(a_3) = -0.5$$

For the purposes of numerical simulation the disturbances were generated by $\xi_0(t) = 0.16\ G(0, 1)$ and $\eta(r_i, t) = 0.5\ G(0, 1)$, where $G(0, 1)$ is a normally distributed random variable with zero mean and unit standard deviation.

The control scheme of the last section was implemented in a numerical experiment. Equation (14) was solved by the alternating direction method. $\delta(s)$ was approximated by $1/\Delta r$, Δr being the spatial mesh size, $Q = 4/3$, and $R_0^{-1} = 0.0256$. The control law of equation (26) is in this case

$$v(a_i, t) = \begin{cases} v^*(a_i) & \hat{x}(a_i, t) < x^d(a_i) \\ v_*(a_i) & \hat{x}(a_i, t) > x^d(a_i) \end{cases} \tag{32}$$

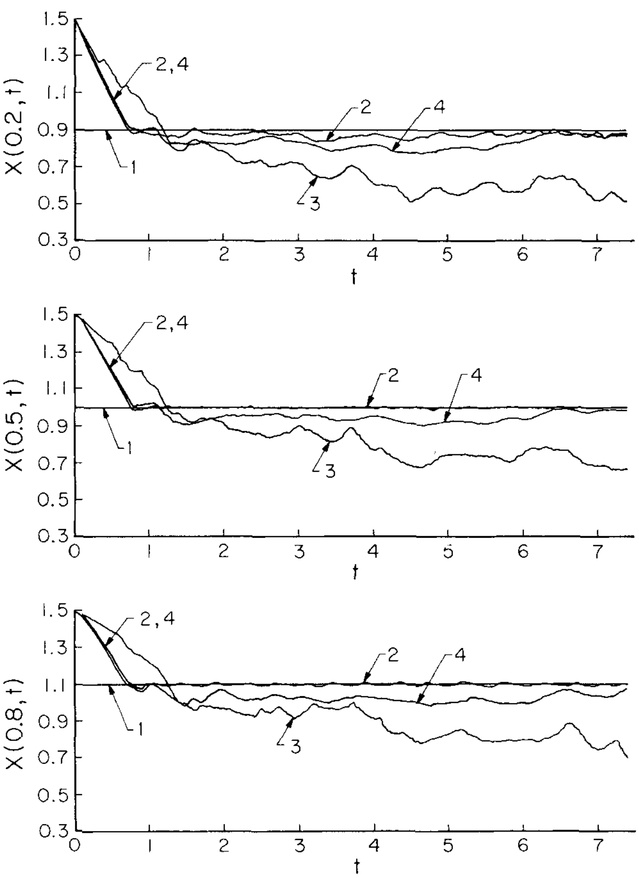

Fig. 2. Comparison of suboptimal control performance: Curve 1—desired state; Curve 2—perfect observations; Curve 3—noisy observations, no filter; Curve 4—noisy observations, no filter.

The initial conditions for Equations (11) and (14) were $\hat{x}(r, 0) = 2.0$ and $P(r, s, 0) = 3.0$.

The results are shown in Figures 1 and 2 for $t_f = 7.4$. Figure 1 shows the performance of the filter. The two curves at each of the three measurement locations are the true state and the filter estimate. Figure 2 shows a comparison of the performance of the control policy of Equation (32) in the presence and absence of the filter. With no filter $\hat{x}(a_i, t)$ in Equation (32) is replaced by the observation $y(a_i, t)$. Without the filter the suboptimal control law is ineffective due, of course, to the noise in the observations.

SUMMARY

We have presented a practical suboptimal scheme for the volume control of the system (1) to (4) based on minimizing an instantaneous performance index. The performance of both the filter and the control scheme was demonstrated for a parabolic system. The suboptimal control problem in which boundary controls exist has yet to be solved. Since this is an important class of distributed control systems it is worthy of future attention. Finally, no general results on the stability of suboptimal schemes such as that proposed here appear to be attainable. This is an important issue which should be studied computationally with a system model prior to implementation.

LITERATURE CITED

Bensoussian, A., "On the Separation Principle for Distributed Parameter Systems," *Proc. IFAC Symp. Control of Distributed Parameter Systems,* Banff (1971).

Carrier, G. F., M. Krook, and C. E. Pearson, *Functions of a Complex Variable; Theory and Technique,* McGraw-Hill, New York (1966).

Hwang, M., J. H. Seinfeld, and G. R. Gavalas, "Optimal Least-Square Filtering and Interpolation in Distributed Parameter Systems," *J. Math. Anal. Appl.,* 39, 1, 49 (1972).

Kushner, H. J., "On the Optimal Control of a System Governed by a Linear Parabolic Equation with White Noise Inputs," *SIAM J. Control,* 6, 4 (1968).

Paradis, W. O., and D. Perlmutter, "Optimality and Computational Feasibility in Transient Control, Part I, A Modified Criterion for Optimality," *AIChE J.,* **12,** 876 (1966).

Pell, T. M., Jr., and R. Aris, "Some Problems in Chemical Reactor Analysis with Stochastic Features, II. Control of Linearized Distributed Systems on Discrete and Corrupted Observations," *Ind. Eng. Chem. Fundamentals,* 9, 15 (1970).

Seinfeld, J. H., "Optimal Stochastic Control of Nonlinear Systems," *AIChE J.,* **16,** 1016 (1970).

———., G. R. Gavalas, and M. Hwang, "Nonlinear Filtering in Distributed Parameter Systems," *J. Dyn. Syst. Meas. Control,* 93G, 157 (1971).

Sholar, M. S., and D. M. Wiberg, "Canonical Equations for Boundary Feedback Control of Stochastic Distributed Parameter Systems," *Automatica,* **8,** 287 (1972).

Thau, F. E., "Optimum Control of a Class of Randomly Excited Distributed Parameter Systems," *Proc. Fourth IFAC Congr.,* Warsaw, Poland (1969).

Tzafestas, S. G., and J. M. Nightingale, "Optimal Control of a Class of Linear Stochastic Distributed-Parameter Systems," *Proc. IEEE,* **115,** 1213 (1968).

Yu, T. K., and J. H. Seinfeld, "Control of Stochastic Distributed Parameter Systems," *J. Optimiz. Theory Appl.,* **10,** 362 (1972).

Vermeychuk, G., and L. Lapidus, "Suboptimal Feedback Control of Distributed Systems," *AIChE J.,* **19,** 123 (1973).

Manuscript received October 16, 1972; note accepted November 22, 1972.